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The Error Behaviour of Collocation and Galerkin Methods in Solving Integral Equations

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Abstract

This paper was devoted to study the error behaviour of the solutions for Fredholm and Volterra integral equations of the second kind using Collocation and Galerkin methods at N=3:100. This paper started with an introduction to show the related work. In addition, we presented the analysis of the numerical methods which we used. Under certain conditions, Banach's fixed point theorem was used to prove the existence and uniqueness for the error integral equation. We presented a comparison between the maximum and minimum errors obtained by Collocation and Galerkin methods. Moreover, some applications were given to satisfy our study. Results were represented in groups of tables and figures.

Keywords: Integral Equations (IEs), Collocation method (CM), Galerkin method (GM), Behaviour of error.

1. Introduction

Many physical problems are modeled in the form of IEs [2, 6], so solving IEs plays an important role in many applications. There are some books present different methods to solve them [2, 3, 5, 6, 14, 17, 18]. In the last two decades, many authors devoted many articles to solve IEs numerically. Kotsireas [7] presented a survey on solution methods for IEs. Mandal and Bhattacharya [9] obtained Appr. solutions of some classes of FIEs, using Bernstein polynomials as basis. Rahman [13] discussed numerical solutions of VIEs, using GM with Hermite polynomials. Numerical treatment for FIE of the second kind is devoted by Rihan [15]. Qatanani [16] discussed analytical and numerical solutions of VIES of the second kind. Mamadu and Njoseh [8] presented numerical technique to solve VIEs, using GM with orthogonal polynomials. Babasola and Irakoze [4] provided CM for numerical solutions of FIEs with certain orthogonal basis functions in interval [0,1]. Also, Aigo used SR and TR for solving VIEs of the second kind [1]. Nadir and Rahmoune [12] presented a numerical method to solve VIEs of the second kind based on the adaptive SR. A modified SR for solving FIEs of the second kind is discussed by Mirzaee and Piroozfar [10]. Therefore, Mirzaee [11] investigated a numerical method for solving FIEs of second kind based on SR.

This paper is devoted to study the stability of error of some numerical methods for solving FIEs and VIEs. We concentrate our interest on using CM and GM. Also, we studied the behavior of errors at fixed points in each case, investigating the Max. and Min. error at each point and corresponding N functions.

2. Numerical Methods

There are many numerical techniques for solving FIES and VIES of the second kind [2, 3, 5, 6, 18]. So, we concentrate our interest to use CM and GM**.**

Solving FIEs and VIEs using CM

Consider the following FIE of the second kind

$$\mu\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt \qquad (2.1)$$

where $x = x(x_1, x_2, \ldots, x_n)$ and μ and λ are constants which has a unique solution under the following conditions:

- The known function $f(x)$ is continuous in the space $L_2([a,b]\times[a,b])$ and its norm is defined as $\| f(x) \| = \left[\int_a^b |f(x)|^2 dx\right]^{\frac{1}{2}} \leq A$.
- The kernel $k(x,t) \in L_2([a,b]\times[a,b]) \quad \forall\ x,t \in [a,b]$ and satisfies $|k(x,t)| \leq B$,

where A and B are constants.

Using CM in Eq.(2.1), we get

$$\mu S_N(x) = f(x) + \lambda \int_a^b k(x,t) S_N(t)dt + E(x, c_1, c_2, \ldots, c_N) \qquad (2.2)$$

where

$$S_N(x) = \sum_{i=1}^N c_i\psi_i, \qquad (2.3)$$

c_i are unknown constants and ψ_i are linearly independent functions.

The error in Eq.(2.2) vanishes at N points $x_1, x_2, \ldots, x_N$, so Eq.(2.2) becomes

$$E(x_i, c_1, c_2, \ldots, c_N) = \mu S_N(x_i) - f(x_i) - \lambda \int_a^b k(x_i,t) S_N(t)dt, \quad 1 \leq i \leq N. \qquad (2.4)$$

Also, for the following VIE of the second kind

$$\mu\phi(x) = f(x) + \lambda \int_a^x k(x,t)\phi(t)dt. \qquad (2.5)$$

If the kernel $k(x,t)$ is continuous in the interval $0 \leq t \leq x \leq X$ and the known function $f(x)$ also continuous in the interval $0 \leq x \leq X$, then Eq.(2.5) has a unique solution under the following conditions:

- The known function $f(x)$ is continuous in the Space $C[a,X], \forall x \in [a,X]$ and satifies $|f(x)| \leq C$.
- The kernel $k(x,t) \in C[a,X],\ x,t \in [a,X]$ and satisfies $|k(x,t)| < D$.

Where C and D are constants.

Applying CM to Eq.(2.5) yields

$$\mu S_N(x) = f(x) + \lambda \int_a^x k(x,t) S_N(t)dt + E(x, c_1, c_2, \ldots, c_N). \qquad (2.6)$$

Insisting the error in Eq.(2.6) vanishes at N points $x_1, x_2, \ldots, x_N$, Eq.(2.6) becomes

$$E(x_i, c_1, c_2, \dots, c_N) = \mu S_N(x_i) - f(x_i) - \lambda \int_a^{x_i} k(x_i, t) S_N(t) dt, \qquad 1 \le i \le N. \tag{2.7}$$

Solving FIEs and VIEs using GM

Eq.(2.1) can be written as

$$E(x, c_1, c_2, \dots, c_N) = \mu S_N(x) - f(x) - \lambda \int_a^b k(x,t) S_N(t) dt. \tag{2.8}$$

From GM, the error is orthogonal to N-linearly independent functions $\chi_1(x), \chi_2(x)$, $\chi_3(x), \dots, \chi_N(x)$ on the interval $[a, b]$. So, we have

$$\int_a^b \chi_j(y) E(y, c_1, c_2, \dots, c_N) dy = 0. \tag{2.9}$$

Substation from Eq.(2.3), Eq.(2.8) and Eq.(2.9) yield

$$\int_a^b \chi_j(y) \left[\mu \sum_{i=1}^{N} c_i \psi_i(y) - \lambda \int_a^b k(y,t) \sum_{i=1}^{N} c_i \psi_i(t) dt \right] dy = \int_a^b \chi_j(y) f(y) dy, 1 \le j \le N. \tag{2.10}$$

For the VIE of the second kind, Eq.(2.6) yields

$$E(x, c_1, c_2, \dots, c_N) = \mu S_N(x) - f(x) - \lambda \int_0^x k(x,t) S_N(t) dt. \tag{2.11}$$

Using GM, we get

$$\int_a^X \chi_j(y) \left[\mu S_N(y) - \lambda \int_a^x k(y,t) S_N(t) dt \right] dy = \int_a^X \chi_j(y) f(y) dy, \qquad 1 \le j \le N. \tag{2.12}$$

From Eq.(2.3), Eq.(2.12) can be written in the form

$$\int_a^X \chi_j(y) \left[\mu \sum_{i=1}^{N} c_i \psi_i(y) - \lambda \int_a^x \left(k(y,t) \sum_{i=1}^{N} c_i \psi_i(t) \right) dt \right] dy = \int_a^X \chi_j(y) f(y) dy, \quad 1 \le j \le N. \tag{2.13}$$

3. Stability of the error

Assume that the solution of FIE (2.1) is

$$\phi(x) = \sum_{i=0}^{\infty} \psi_i(x) \tag{3.1}$$

and $\quad \phi_N(x) = \sum_{i=0}^{N} \psi_i(x)$ (3.2)

is the Appr. solution of Eq.(2.1), we have

$$\phi_N(x) = \frac{1}{\mu} f(x) + \frac{\lambda}{\mu} \int_a^b k(x,t) \phi_N(t) dt. \tag{3.3}$$

Eq.(2.1) and Eq.(3.3) yield the error $E(x)$ s.t.

$$E(x) = \phi(x) - \phi_N(x) = \frac{\lambda}{\mu} \int_a^b k(x,t)(\phi(t) - \phi_N(t)) dt. \tag{3.4}$$

Eq.(3.4) becomes

$$E(x) = \frac{\lambda}{\mu} \int_a^b k(x,t) E(t) dt. \tag{3.5}$$

Eq.(3.5) represents IE of the error $E(x)$ with the same kernel of FIE (2.1).

Theorem 3.1

If $k(x,t)$ is continuous in $a \le t \le x \le b$ and satisfies $|k(x,t)| \le A$, then Eq.(3.5) has a unique continuous solution in $a \le x \le b$ under the condition

$$|\lambda| < \frac{|\mu|}{A(b-a)}.$$

Proof Now, we are going to prove the normality and continuity of Eq.(3.5) which can be written in the form of integral operator

$$V_E = \frac{\lambda}{\mu} \int_a^b k(x,t) E(t) dt. \tag{3.6}$$

From the normality, we get

$$\begin{aligned} \| V_E \| &= \left\| \frac{\lambda}{\mu} \int_a^b k(x,t) E(t) dt \right\| \\ &\le |\frac{\lambda}{\mu}| \left\| \int_a^b |k(x,t)| E(t) dt \right\| \\ &\le |\frac{\lambda}{\mu}| A \| E(t) \| \left\| \int_a^b dt \right\| \\ &\le |\frac{\lambda}{\mu}| A(b-a) \| E(t) \| \\ &\le \varrho \| E \|, \qquad \varrho = |\frac{\lambda}{\mu}| A(b-a). \end{aligned} \tag{3.7}$$

Since V is contracting operator, we obtain $\varrho < 1$. So, we get

$$|\lambda| < \frac{|\mu|}{A(b-a)}. \tag{3.8}$$

Therefore, the integral operator V has a normality.

Assume that the two functions $E_1(x)$ and $E_2(x)$ in the space $L_2[a, b]$ satisfy Eq.(3.6), then we have

$$V_{E_1} = \frac{\lambda}{\mu} \int_a^b k(x,t) E_1(t) dt \text{ and } V_{E_2} = \frac{\lambda}{\mu} \int_a^b k(x,t) E_2(t) dt. \tag{3.9}$$

So, we get

$$V_{E_2} - V_{E_1} = \frac{\lambda}{\mu} \int_a^b k(x,t)(E_2(t) - E_1(t)) dt. \tag{3.10}$$

From the properties of the norm of $L_2[a, b]$, we obtain

$$\| V_{E_2} - V_{E_1} \| = \left\| \frac{\lambda}{\mu} \int_a^b k(x,t) \big(E_2(t) - E_1(t)\big) dt \right\| \le |\frac{\lambda}{\mu}| A(b-a) \| E_2(t) - E_1(t)) \|. \tag{3.11}$$

Hence, we have

$$\| V_{E_2} - V_{E_1} \| \le \varrho \| E_2 - E_1 \|, \quad \varrho < 1 \tag{3.12}$$

with $\quad |\lambda| < \frac{|\mu|}{A(b-a)}$. (3.13)

The inequality (3.12) leads to the continuity of the integral operator V. Hence, V is contraction operator in the space $L_2[a, b]$, so Banach's fixed point theorem yields that V has a unique fixed point which means that Eq.(3.5) has a unique solution.

Similarly, for VIE of the second kind, we obtain

$$E(x) = \frac{\lambda}{\mu} \int_a^x k(x,y) E(y) dy, \tag{3.14}$$

Eq.(3.14) represents IE of the error $E(x)$ with the same kernel of VIE (2.5).

Theorem 3.2

If $k(x,t) \in C[a, X]$ and is continuous $\forall x,\ t \in [a, X]$ satisfying $|k(x,t)| \le A$, then Eq.(3.14) has a unique continuous solution under the condition

$$|\lambda| < \frac{|\mu|}{A(X-a)}.$$

Proof Now, we prove the normality and continuity of Eq.(3.14) which can be written in the form of integral operator

$$W_E = \frac{\lambda}{\mu} \int_a^x k(x,t) E(t) dt. \tag{3.15}$$

From the normality, we get

$$\begin{aligned}\| W_E \| &= \left\| \frac{\lambda}{\mu} \int_a^x k(x,t)E(t)dt \right\| \\ &\leq |\frac{\lambda}{\mu}| \left\| \int_a^x |k(x,t)| E(t)dt \right\| \\ &\leq |\frac{\lambda}{\mu}| A \| E(t) \| \left\| \int_a^x dt \right\| \qquad (3.16) \\ &\leq |\frac{\lambda}{\mu}| A(X-a) \| E(t) \| \\ &\leq \eta \| E \|, \qquad \eta = |\frac{\lambda}{\mu}| A(X-a).\end{aligned}$$

Since W is contracting operator, we obtain $\eta < 1$. So, we get

$$|\lambda| < \frac{|\mu|}{A(X-a)}. \qquad (3.17)$$

Therefore, the integral operator W has a normality.

Assume that the two functions $E_1(x)$ and $E_2(x)$ in the space $C[a,X]$ satisfy Eq.(3.15), then we have

$$W_{E_1} = \frac{\lambda}{\mu} \int_a^x k(x,t)E_1(t)dt$$

$$\text{and} \quad W_{E_2} = \frac{\lambda}{\mu} \int_a^x k(x,t)E_2(t)dt. \qquad (3.18)$$

So, we get

$$W_{E_2} - W_{E_1} = \frac{\lambda}{\mu} \int_a^x k(x,t)E_2(t) - E_1(t))dt. \qquad (3.19)$$

From the properties of the norm, we obtain

$$\begin{aligned}\| W_{E_2} - W_{E_1} \| &= \left\| \frac{\lambda}{\mu} \int_a^x k(x,t)(E_2(t) - E_1(t))dt \right\| \\ &\leq |\frac{\lambda}{\mu}| A(X-a) \| E_2(t) - E_1(t)) \|. \qquad (3.20)\end{aligned}$$

So, we get

$$\| W_{E_2} - W_{E_1} \| \leq \eta \| E_2 - E_1 \|, \quad \eta < 1 \qquad (3.21)$$

$$\text{with} \qquad |\lambda| < \frac{|\mu|}{A(X-a)}. \qquad (3.22)$$

The inequality (3.21) leads to the continuity of the integral operator W. W is contraction operator in the space $C[a,X]$ so, Banach's fixed point theorem yields that W has a unique fixed point which means that Eq.(3.14) has a unique solution.

4. Numerical Results and Discussion

many physical problems studies lead to IE like "Transverse oscillations of a bar" which can be convert to Fredholm or Volterra IE of the second kind depending to the conditions of the problems see [Tricomi. 1985 and Rahman 2007]. Furthermore, the study of "Electric circuits" can be reduced to FIE see [Stewart, 2015]. So, we are going to focus on studying the behaviour of the error of IEs and compare between the presented methods.

Example 4.1 Consider the following FIE of the second kind

$$\phi(x) = f(x) + \int_0^\pi 2x^2 t\phi(t)dt, \qquad \phi(x) = \sin(x). \qquad (4.1)$$

Applying CM and GM, choosing the expansion of Appr. solution as a sum of sinusoidal functions

$$S_N(x) = \sum_{k=1}^{N} c_k \left(\left| \sin\left(\frac{k\pi}{2}\right) \right| \cos(\frac{k+1}{2}x) + \left| \sin\left(\frac{k+1}{2}\pi\right) \right| \sin(\frac{k}{2}x) \right).$$

Firstly, it can be observed that using CM the error is increasing through increasing N as shown in Fig (4.1). When we take the Appr. solution in the form of five terms "$N = 5$", the Max. value error value is $(7.73579 \times 10^{-16})$ at $x = \pi$. Also, in Fig (4.3), the error closed to zero except at $N = 9$ it is increasing to touch the highest value $(1.18484 \times 10^{-14})$ at $x = \pi$. Fig (4.5) shows that the highest value of the error is $(7.53942 \times 10^{-11})$ at $x = \pi$ with $N = 17$. Fig (4.7) shows that the behaviour of the error is stable but at $N = 21$ the error reaches to peak at $x = \pi$ with value (4.55044×10^{-9}).

Secondly, using GM shows that the error is increasing due to increasing N, see Fig (4.2), Fig (4.4) and Fig (4.6). The Max. error in Fig.(4.2) is $(1.11022 \times 10^{-16})$ at $x = \frac{3\pi}{4}$ with $N = 5$, in Fig (4.4) is $(3.8256{,}9 \times 10^{-15})$ at $x = 0$ with $N = 10$ and in Fig (4.6) is $(6.63085 \times 10^{-11})$ at $x = 0$ with $N = 10$. In Fig (4.8) the behaviour of the error is stable but at $N = 82$, the error reaches to the peak at $x = \pi$ with value (1.8135×10^{-4}).

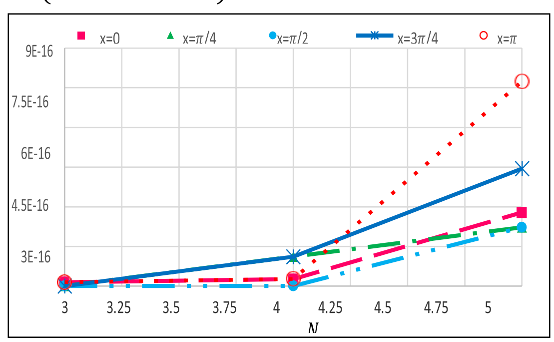

Fig (4.1): The behavior of the error at different values of N = 3 : 5 using CM.

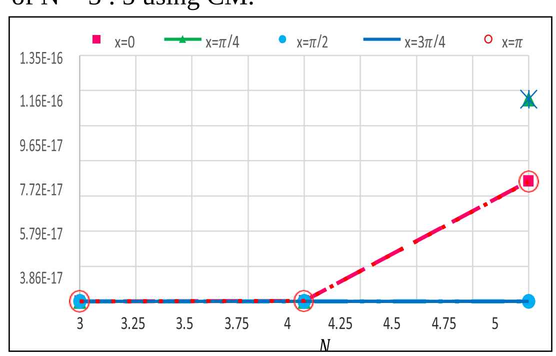

Fig (4.2): The behavior of the error at different values of N = 3:5 using GM.

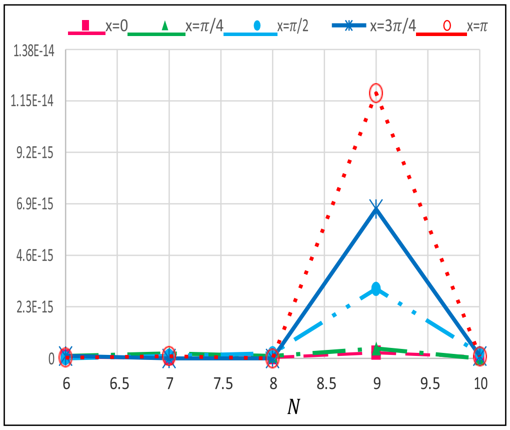

Fig (4.3): The behavior of the error at different values of N = 6 : 10 using CM.

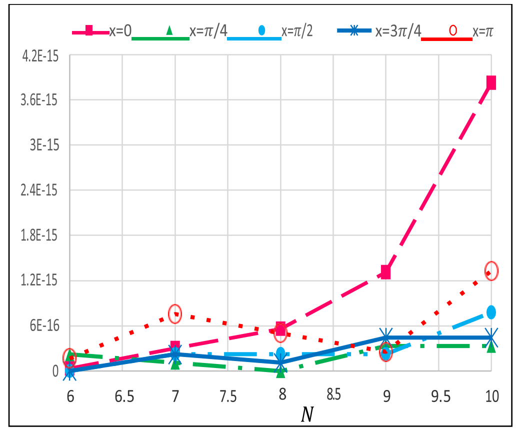

Fig (4.4): The behavior of the error at different values of N = 6 : 10 using GM.

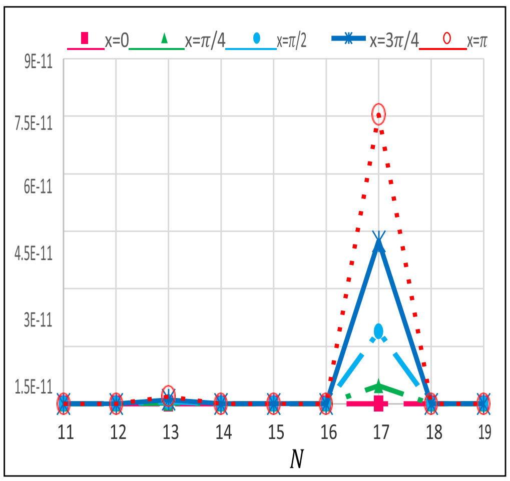

Fig (4.5): The behavior of the error at different values of N = 11 : 19 using CM.

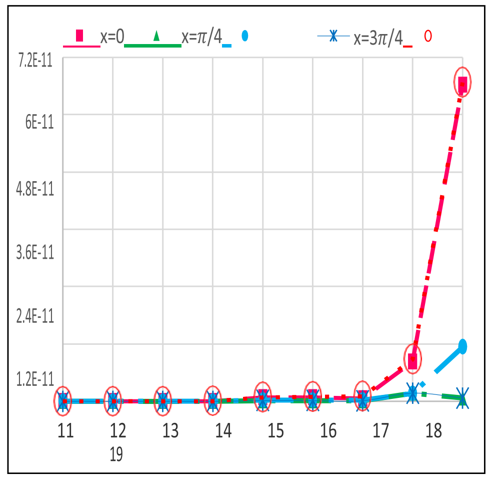

Fig (4.6): The behavior of the error at different values of N = 11 : 19 using GM.

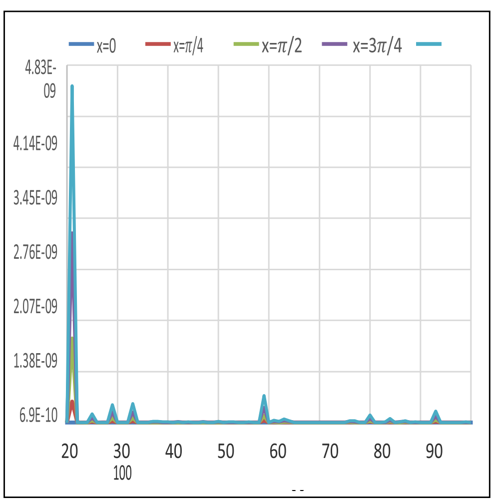

Fig (4.7): The behavior of the error at different values of $N = 20{:}\,100$ using CM

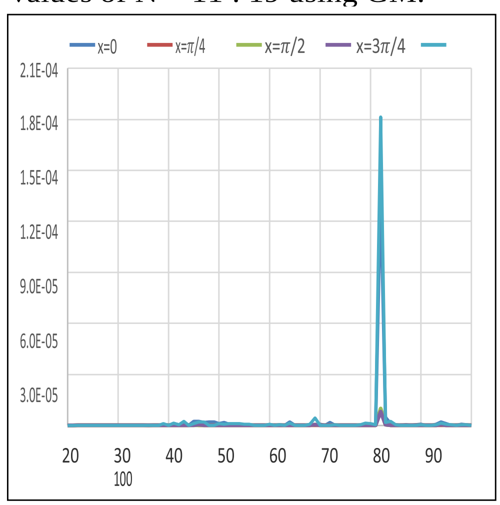

Fig (4. 8): The behavior of the error at different values of $N = 20{:}\,100$ using GM

Table (1): A comparison between Max. and Min. errors using CM, GM in example (4.1).

x_i	Max. error				Min. error			
	CM		GM		CM		GM	
	E_{max}	N	E_{max}	N	E_{min}	N	E_{min}	N
0	7.99E-15	93	0.00014	82	0	91	9.62E-22	3
$\frac{\pi}{4}$	2.84E-10	21	8.75E-06	82	0	3,10,11,23	0	3, 4, 8
$\frac{\pi}{2}$	1.14E-09	21	9.95E-06	82	0	3,4,6,11,14,16,18	0	3,4,5,6
$\frac{3\pi}{4}$	2.56E-09	21	8.11E-06	82	0	3,7,8,12	0	3,4,6
π	4.55E-09	21	0.000181	82	1.35E-18	8	9.62E-22	3

Table (2): A comparison between error obtained by CM and GM at five points for different N functions in example (4.1).

x_i	N	Exact	CM_{Error}	GM_{Error}
0	3	0	1.42042E-17	9.62322E-22
	4		2.66782E-17	2.03423E-19
	5		2.78443E-16	6.60252E-17
	.		.	.
	100		6.41848E-16	2.94777E-08
$\frac{\pi}{4}$	3	0.707106781	0	0
	4		1.11022E-16	0
	5		2.22045E-16	1.11022E-16
	.		.	.
	100		1.23568E-13	1.43646E-09
$\frac{\pi}{2}$	3	1	0	0
	4		0	0
	5		2.22045E-16	0
	.		.	.
	100		4.96048E-13	8.19014E-10
$\frac{3\pi}{4}$	3	0.707106781	0	0
	4		1.11022E-16	0
	5		4.44089E-16	1.11022E-16
	.		.	.
	100		1.11844E-12	2.82912E-09
π	3	0	1.42042E-17	9.62322E-22 5
	4		2.66782E-17	2.03423E-19
	5		7.73579E-16	6.5776E-17
	.		.	.
	100		1.98807E-12	4.35455E-08

Example 4.2 Consider the following FIE of the second kind

$$\phi(x) = f(x) + \int_0^1 (3t + x)\phi(t)dt,\ \phi(x) = \frac{1-e^{-x}}{1+e^{x}}. \quad (4.2)$$

Applying CM and GM in this example, choosing the expansion of Appr. solution as a sum of exponential functions s.t.

$$S_N(x) = \sum_{k=1}^{N} c_k e^{(k-1)x}, \qquad N = 3,4,\ldots,100.$$

Firstly, using CM, as a general trend, the error is decreasing due to increasing N and the Max. error value (2.79×10^{-2}) is obtained at $x = 0.25$ with $N = 3$. Also, the Min. value $(1.90184 \times 10^{-12})$ is obtained at $x = 1$ with $N = 44$.

Secondly, using GM a glance at the graphs revels that owing to increasing N the error is dramatically decreasing as show in Fig (4.10) and Fig (4.12) but in Fig (4.14) and Fig (4.16), it changes sporadically hitting the peak (2.55×10^{-4}) at $x = 1$ with $N = 79$.

Table (3): A comparison between Max. and Min. errors using CM, GM in example (4.2).

x_i	Max. error				Min. error			
	CM		GM		CM		GM	
	E_{max}	N	E_{max}	N	E_{min}	N	E_{min}	N
0	0.00633	3	0.023654	3	6.77E-12	66	4.77E-09	26
0.25	0.02791	3	0.008722	3	1.3E-11	66	2.69E-10	50
0.5	0.01134	3	0.002812	3	3.04E-12	66	3.85E-09	80
0.75	0.00889	3	0.008948	3	1.27E-11	66	2.12E-08	21
1	0.01636	3	0.018479	3	1.9E-12	44	1.43E-08	43

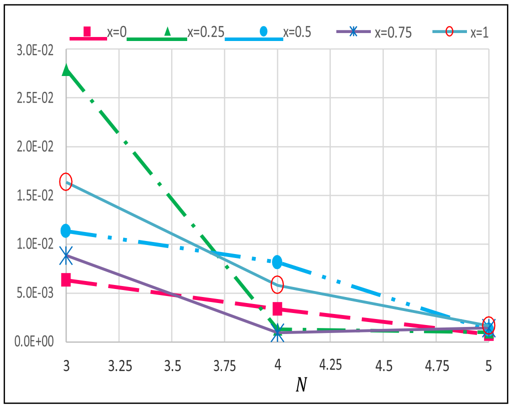

Fig. 4.9: The behavior of the error at different values of N = 3 : 5 using CM.

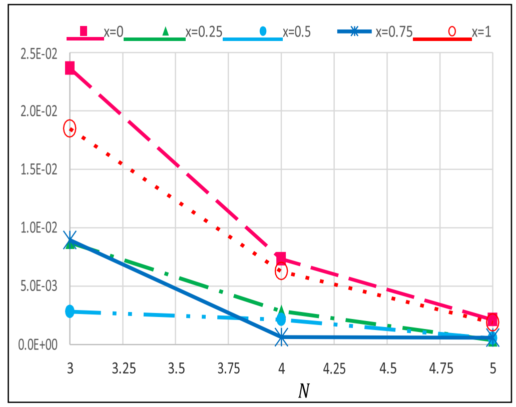

Fig. 4.10: The behavior of the error at different values of N = 3 : 5 using GM.

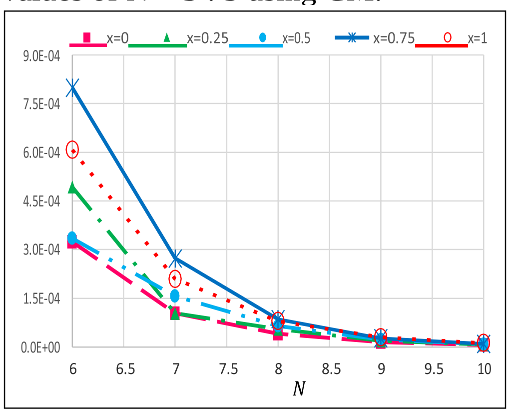

Fig (4.11): The behavior of the error at different values of N = 6 : 10 using CM.

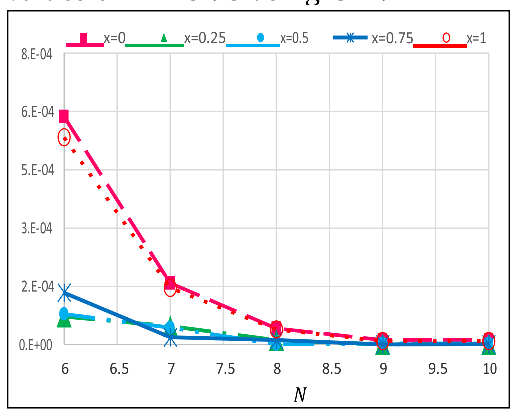

Fig (4.12): The behavior of the error at different values of N = 6 : 10 using GM.

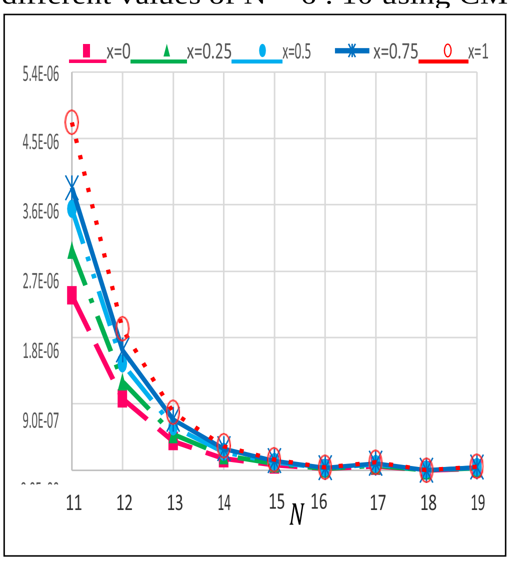

Fig (4.13): The behavior of the error at different values of N = 11 : 19 using CM.

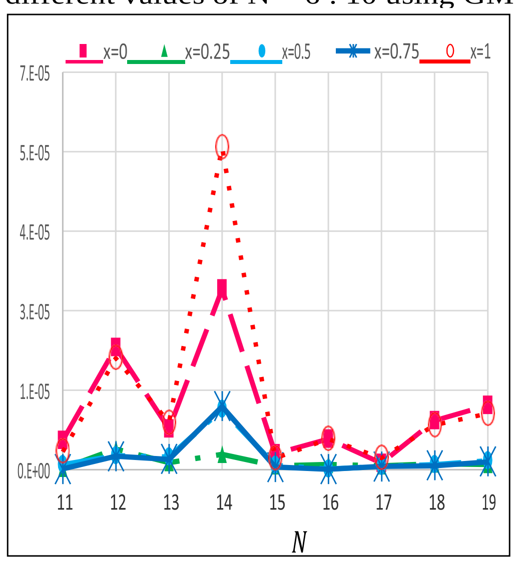

Fig (4.14): The behavior of the error at different values of N = 11 : 19 using GM.

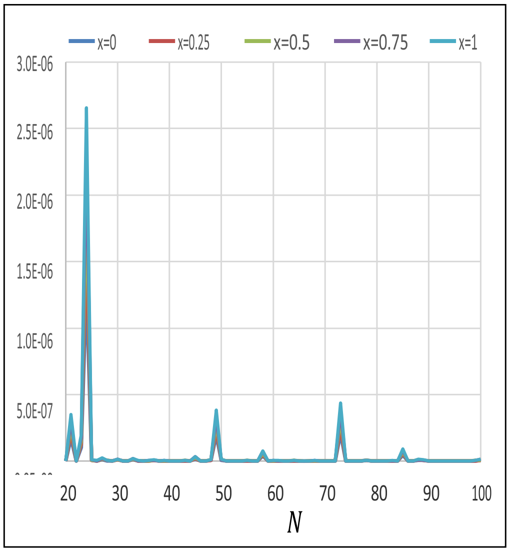

Fig (4.15): The behavior of the error at different values of N = 20 : 100 using CM.

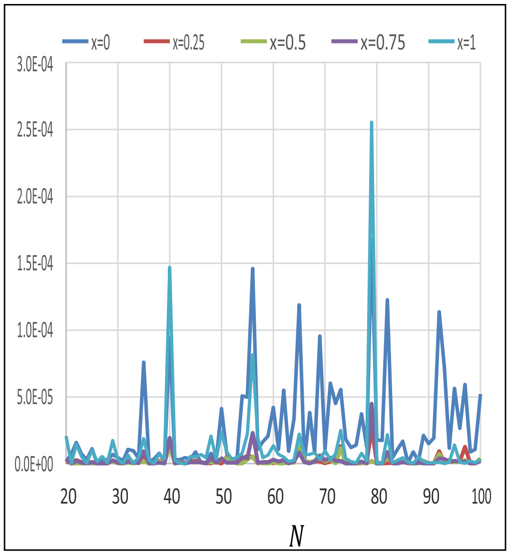

Fig (4.16): The behavior of the error at different values of N = 20 : 100 using GM.

Table (4): A comparison between error obtained by CM and GM at five points for different N functions in example (4.2).

x_i	N	Exact	CM_{Error}	GM_{Error}
0	3	0	0.006321	0.023654
	4		0.003342	0.007343
	5		0.000775	0.002119
	.		.	.
	100		6.95E-09	5.23E-05
0.25	3	0.096846	0.027914	0.008722
	4		0.00128	0.002829
	5		0.000996	0.000354
	.		.	.
	100		6.76E-09	3.56E-06
0.5	3	0.148551	0.01134	0.002812
	4		0.008137	0.002108
	5		0.001217	0.000515
	.		.	.
	100		7.32E-09	4.18E-06
0.75	3	0.169276	0.008868	0.008948
	4		0.000949	0.000644
	5		0.001438	0.00058
	.		.	.
	100		1.74E-08	2.05E-06
1	3	0.170003	0.016358	0.018479
	4		0.005813	0.006258
	5		0.001659	0.001879
	.		.	.
	100		1.61E-08	2.97E-06

Example 4.3 Consider the following VIE of the second kind $\phi(x) = f(x) + \int_0^x 2tx^3\phi(t)dt, \phi(x) = \ln(x^2+1)$. (4.3)

Applying CM and GM, we choose the expansion of Appr. solution as a sum of polynomial functions

$$S_N(x) = \sum_{k=1}^{N} c_k x^{k-1}, \qquad N = 3,4,\dots,100.$$

Firstly, in this example as an overall trend when we use CM the error is decreasing due to increasing N. In Fig (4.17) the error decreases dramatically by increasing N with Max. error value (2.34×10^{-2}) at $x = 0.8$ and $N = 3$. Also, when N increases from $6:100$, the error changes sporadically with Max. value (2.30483×10^{-6}) at $x = 0.2$ and $N = 7$.

Secondly, applying GM the error is decreasing owing to increasing N and the peak value (7.91×10^{-3}) is obtained at $x = 0.8$ with $N = 3$ as shown in Fig (4.18). Also, for $N \geq 11$ the behaviour of the error is erratic with Max. value $(1.87209 \times 10^{-06})$ at $x = 0$ and $N = 89$ as shown in Fig (4.24).

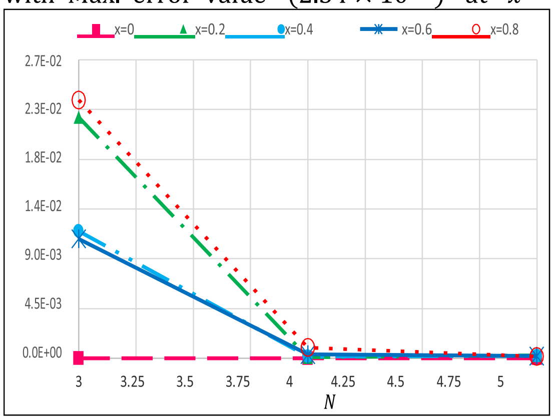

Fig (4.17): The behavior of the error at different values of N = 3 : 5 using CM.

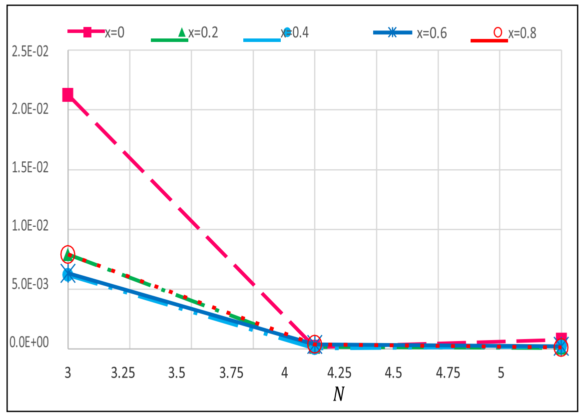

Fig (4.18): The behavior of the error at different values of N = 3 : 5 using GM.

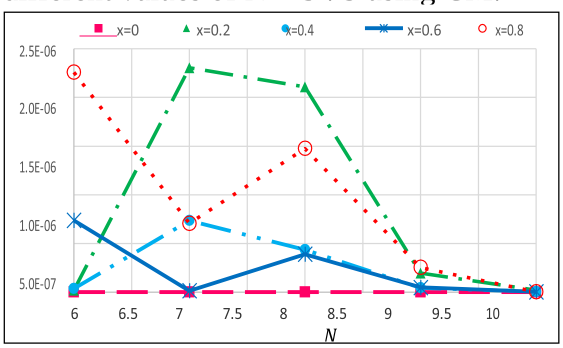

Fig (4.19): The behavior of the error at different values of N = 6 : 10 using CM.

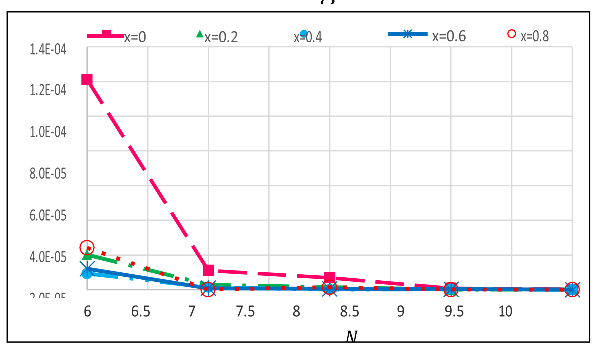

Fig (4.20): The behavior of the error at different values of N = 6 : 10 using GM.

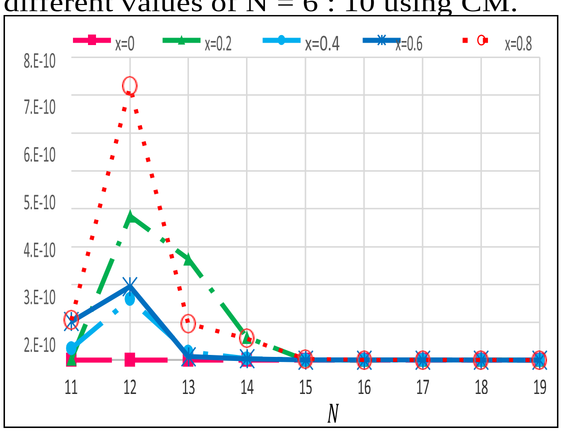

Fig (4.21): The behavior of the error at different values of N = 11 : 19 using CM.

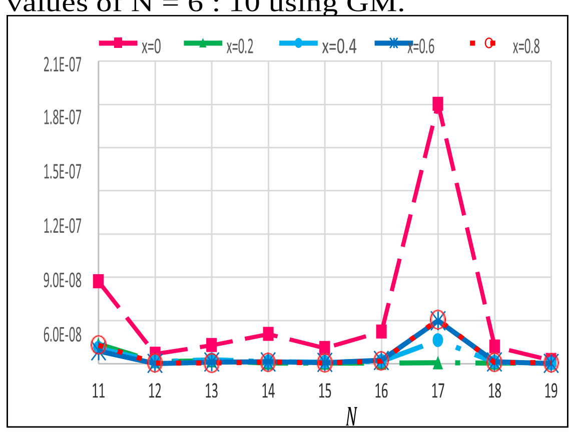

Fig (4.22): The behavior of the error at different values of N = 11 : 19 using GM.

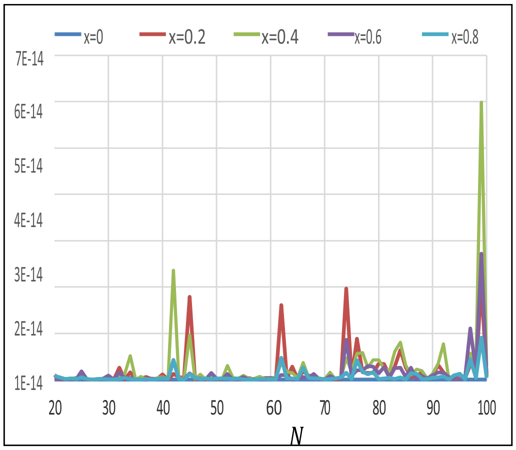

Fig (4.23): The behavior of the error at different values of N = 20 : 100 using CM.

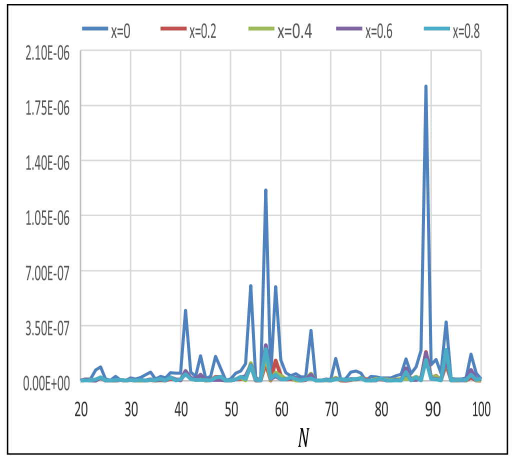

Fig (4.24): The behavior of the error at different values of N = 20 : 100 using GM.

Table (5): A comparison between Max. and Min. errors using CM, GM in example (4.3).

x_i	Max. error				Min. error			
	CM		GM		CM		GM	
	E_{max}	N	E_{max}	N	E_{min}	N	E_{min}	N
0	0	3-100	0.021263	3	0	3-100	3.59E-10	68
0.2	0.021801	3	0.007905	3	0	55	2.17E-11	73
0.4	0.011498	3	0.006177	3	2.78E-17	22,35, 48	8.88E-11	19
0.6	0.010797	3	0.006333	3	0	31,35, 41,70	6.26E-11	23
0.8	0.023365	3	0.007908	3	0	27,36, 44,56, 57,85	2.09E-10	77

Table (6): A comparison between error obtained by CM and GM at five points for different N functions in example (4.3).

x_i	N	Exact	CM_{Error}	GM_{Error}
0	3	0	0	0.021263
	4		0	0.000158
	5		0	0.000784
	.		.	.
	100		0	1.15E-08
0.2	3	0.039221	0.021801	0.007905
	4		7.56E-05	0.000175
	5		0.000274	8.15E-05
	.		.	.
	100		8.26E-16	6.59E-10
0.4	3	0.14842	0.011498	0.006177
	4		0.00016	5.05E-05
	5		0.000232	0.000196
	.		.	.
	100		2.91E-15	2.96E-09
0.6	3	0.307485	0.010797	0.006333
	4		0.000328	0.000369
	5		0.000202	0.000217
	.		.	.
	100		7.77E-16	1.24E-08
0.8	3	0.494696	0.023365	0.007908
	4		0.00095	0.00037
	5		0.000156	0.000117
	.		.	.
	100		3.89E-16	1.44E-08

5. Conclusion

A numerical treatment for FIEs and VIEs of the second kind using CM and GM is presented. Also, the behaviour of the errors in each case is studied with a comparison between the presented methods. Under certain conditions, Banach's fixed point theorem is used to prove the existence and uniqueness for the equation of the error which has the same kernel of the origin IE. Results are represented in groups of figures and tables for determining the Max. and Min. error in each case.

From Table (1); Firstly, applying CM in example (4.1), the least Max. error value is (7.99×10^{-15}) at $x = 0$ with $N = 93$ and the upper Max. error value is (4.55×10^{-9}) at $x = \pi$ with $N = 21$. Moreover, the least Min. error value is (0) at $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ with $N = 3$ and the upper Min. error value is (1.35×10^{-15}) at $x = \pi$ with $N = 8$. Secondly, applying GM in example (4.1), the least Max. error value is (8.11×10^{-6}) at $x = \frac{3\pi}{4}$ with $N = 82$ and the upper Max. error value is (1.81×10^{-4}) at $x = \pi$ with $N = 82$.

In addition, the least Min. error value is (9.62×10^{-22}) at $x = 0, \pi$ with $N = 3$ and the upper Min. value of errors is (0) at $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and $N = 3,4$.

From Table (3); Firstly, applying CM in example (4.2), the least Max. error value is (6.32×10^{-3}) at $x = 0$ with $N = 3$ and the upper Max. error value is (2.79×10^{-2}) at $x = 0.25$ with $N = 3$. Moreover, the least Min. error value is (1.9×10^{-12}) at $x = 1$ with $N = 44$ and the upper Min. error value is (1.3×10^{-11}) at $x = 0.25$ with $N = 66$. Secondly, applying GM in example (4.2), the least Max. error value is (2.81×10^{-3}) at $x = 0.5$ with $N = 3$ and the upper Max. error value is (2.3×10^{-2}) at $x = 0$ with $N = 3$. Moreover, the least Min. value of errors is (2.69×10^{-10}) at $x = 0.25$ with $N = 50$ and the upper Min. error value is (2.12×10^{-8}) at $x = 0.75$ with $N = 21$.

From Table (5); Firstly, applying CM in example (4.3), the least Max. error value is (1.07×10^{-2}) at $x = 0.6$ with $N = 3$ and the upper Max. error value is (2.33×10^{-2}) at $x = 0.8$ with $N = 3$. Moreover, the least Min. value of errors is 0 at $x = 0, 0.2$ with $N = 55$ and the upper Min. error value is (2.78×10^{-17}) at $x = 0.4$ with $N = 22,35,48$. Secondly, applying GM in example (4.3), the least Max. error value is (6.177×10^{-3}) at $x = 0.4$ with $N = 3$ and the upper Max. error value is (2.1263×10^{-2}) at $x = 0$ with $N = 3$. In addition, the least Min. value of errors is (2.17×10^{-11}) at $x = 0.2$ with $N = 73$ and the upper Min. error value is (3.59×10^{-10}) at $x = 0$ with $N = 68$. So, we can conclude that:

1. In sinusoidal function, increasing N, the error is increasing as shown in example (4.1) but in polynomial and exponential functions the error decreases due to increasing N. as shown in examples (4.2), (4.3).
2. For large N, CM is more effective than GM in polynomial and exponential functions.

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